PERGAMON

International Journal of Solids and Structures 37 (2000) 5505–5515

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

On the elastic stability of uniform beams and circular arches under nonconservative loading

F.M. Detinko

3098-G Whisper Lake Lane, Winter Park, FL 32792, USA

Received 19 February 1999; in revised form 20 May 1999

Abstract

The infinitesimal (local) stability of uniform beams and circular arches under the terminal nonconservative loading is studied. The nonlinear equilibrium with large deflections is expressed in a closed form. Then, the equations which govern a free motion about the equilibrium position are constructed. The eigenvalue analysis of the linearized system of differential equations with varying coefficients is performed by the finite element concept and the subdomain collocation method. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Arch; Nonlinear; Stability; Subdomain collocation

1. Introduction

In the recent paper by Vitaliani et al. (1997) the geometrically nonlinear behavior and infinitesimal stability of structures, which can be modeled as an assemblage of 3D curved beams, is analyzed. The finite element formulation allows to calculate the eigenvalues of the loaded structure with and without damping, thus defining the critical value of the load.

This paper is limited to the plane geometrically nonlinear deformation of beams and circular arches of uniform cross section, subjected to terminal nonconservative forces. Under these assumptions the equilibrium position is expressed in a closed form, in terms of the elliptical functions, and the problem of an infinitesimal stability reduced to an eigenvalue analysis of the sixth order system of linear differential equations with varying coefficients. These equations describe small free vibration in the vicinity of the equilibrium position and it is required to find one or two lowest eigenvalues.

For the eigenvalue analysis, the finite element approach and the method of weighted residuals is used. The selection of unity weighting function (the method of subdomain collocation) allows to simplify the

E-mail address: felideti@magicnet.net (F.M. Detinko).

PII: S0020-7683(99)00229-2

assembly procedure and the assembled matrix as compared to Galerkin method. Additionally, the governing differential equations are reduced to the system of the first order so that two node elements and linear interpolation functions are sufficient.

The advantages and disadvantages of the mixed formulation are discussed in Zienkiewicz and Taylor (1989) for a broad class of problems, and the important disadvantage mentioned is the nonsymmetric stiffness matrix. However, for the nonconservative loads considered here, the stiffness matrix is inherently nonsymmetric regardless of the method used.

The accuracy of the proposed approach is demonstrated in an example of the non-self-adjoint equation with constant coefficients. For the case of varying coefficients the results are compared to those obtained by Vitaliani et al. (1997).

2. Equations of motion of a curve beam

Equations of a plane motion of an inextensional curve beam with large deflections and rotations can be written as (Chernykh, 1986; Detinko, 1998)

$$Q' + \gamma' N + S = 0 \tag{1}$$

$$N' - \gamma' Q + T = 0 \tag{2}$$

$$S' + \gamma' T + \lambda^2 \ddot{\gamma} = 0 \tag{3}$$

$$T' - \gamma' S - \lambda^2 \dot{\gamma}\dot{\gamma} = 0 \tag{4}$$

$$Q = M' \tag{5}$$

$$M = -\beta' \tag{6}$$

$$\beta = \gamma - \varphi \tag{7}$$

Here β is the rotation angle (Fig. 1), Q, N are the shear and normal stress resultants referred to the distorted coordinate system (n, t), M is the bending moment, S, T are the normal and tangential components of inertia forces, $\lambda^2 = \mu R^4/EI$, and μ is mass per unit length.

A prime denotes the derivative with respect to the independent variable φ and a dot indicates the time derivative. All forces are dimensionless and related to the physical forces $(F_n, F_t, M_y, G_n, G_t)$ by

$$(Q, N) = \left(R^2/EI\right)(F_n, F_t), \quad M = (R/EI)M_y, \quad (S, T) = \left(R^3/EI\right)(G_n, G_t) \tag{8}$$

Eqs. (1)–(7), together with initial and boundary conditions, determine the forces and rotation. When the latter is known the displacements can be found as follows. The (x, z) components of displacement (u_x, u_z), normalized with respect to R, are

$$u_x = x - x_0, \qquad u_z = z - z_0 \tag{9}$$

where (x_0, z_0) and (x, z) are coordinates of the point before and after the deformation, respectively. Differentiating (9) and taking into account the geometrical relationships

$$x' = \sin\gamma, \qquad z' = \cos\gamma \tag{10}$$

we obtain

$$u_x' = \sin\gamma - \sin\varphi, \qquad u_z' = \cos\gamma - \cos\varphi \tag{11}$$

The (n, t) components of displacement are

$$w = u_x \cos\gamma - u_z \sin\gamma, \qquad v = u_x \sin\gamma + u_z \cos\gamma \tag{12}$$

Finally, the inertia forces can be expressed in terms of displacements. To this end observe that

$$S = -\lambda^2(\ddot{u}_x \cos\gamma - \ddot{u}_z \sin\gamma), \qquad T = -\lambda^2(\ddot{u}_x \sin\gamma + \ddot{u}_z \cos\gamma) \tag{13}$$

solve Eq. (12) for (u_x, u_z)

$$u_x = w\cos\gamma + v\sin\gamma, \qquad u_z = -w\sin\gamma + v\cos\gamma \tag{14}$$

calculate the second time derivatives and obtain from Eq. (13)

$$S = -\lambda^2\left(\ddot{w} + 2\dot{\gamma}\dot{v} - \dot{\gamma}^2 w + \ddot{\gamma} v\right)$$

$$T = -\lambda^2\left(\ddot{v} - 2\dot{\gamma}\dot{w} - \dot{\gamma}^2 v - \ddot{\gamma} w\right) \tag{15}$$

Notice, that if inertia forces (S, T) form (1) and (2) are inserted into (3) and (4) one can restore Eqs. (35) and (36) of Simmonds (1979).

For the initially straight beam one should set $\varphi = 0$ in Eqs. (7) and (11), and replace the independent variable by $x = s/L$, where L is the length of a beam.

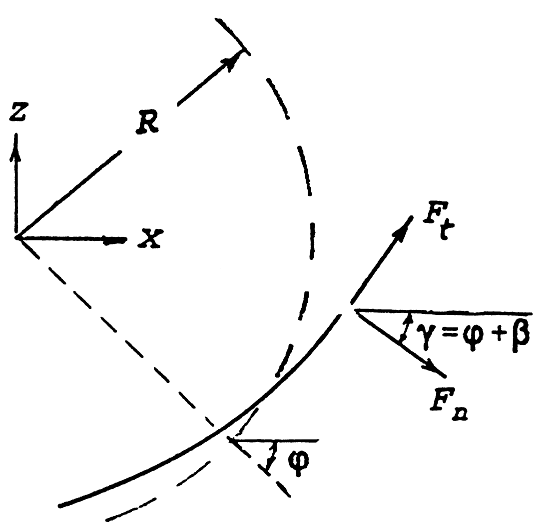

Fig. 1. Geometry and notations.

3. Equilibrium state

At the equilibrium state $\dot{\gamma} = S = T = 0$, Eqs. (3) and (4) are identically satisfied, and Eqs. (5)–(7) yields $Q = -\gamma''$.

Inserting this into Eq. (2) and integrating one obtains the normal force

$$N = C - \Gamma^2/2, \quad \Gamma = \gamma' \tag{16}$$

where C is an integration constant. Eq. (1) now yields

$$\Gamma'' - C\Gamma + \frac{1}{2}\Gamma^3 = 0 \tag{17}$$

Thus, all forces are expressed in terms of the curvature, and the displacements can be found from Eqs. (11) and (12) by direct integration. Depending on the boundary conditions, one of the four following solutions of Eq. (17) may be convenient:

$$\Gamma = \Gamma_0 \mathrm{cn}(h\varphi + \tau), \quad \Gamma_0^2 = 4k^2h^2, \quad C = (2k^2 - 1)h^2 \tag{18}$$

$$\Gamma = \Gamma_0 \frac{\mathrm{sn}(h\varphi + \tau)}{\mathrm{dn}(h\varphi + \tau)}, \quad \Gamma_0^2 = 4k^2(1 - k^2)h^2, \quad C = (2k^2 - 1)h^2 \tag{19}$$

$$\Gamma = \Gamma_0 \mathrm{dn}(h\varphi + \tau), \quad \Gamma_0^2 = 4h^2, \quad C = (2 - k^2)h^2 \tag{20}$$

$$\Gamma = \frac{\Gamma_0}{\mathrm{dn}(h\varphi + \tau)}, \quad \Gamma_0^2 = 4(1 - k^2)h^2, \quad C = (2 - k^2)h^2 \tag{21}$$

where $\mathrm{sn}(x)$, $\mathrm{cn}(x)$, $\mathrm{dn}(x)$ are Jacoby elliptic functions with modulus k. The constants h, τ, k are to be found from the boundary conditions.

4. Stability analysis

To examine the stability of an equilibrium state of the arch by the dynamic method one needs equations, describing the motion of the arch about this state. Introduce small perturbations γ_p, β_p, M_p, N_p, Q_p, S_p, T_p and let

$$\gamma = \gamma_\mathrm{p} + \gamma_s, \quad \beta = \beta_\mathrm{p} + \beta_s, \quad M = M_\mathrm{p} + M_s, \quad N = N_\mathrm{p} + N_s, \quad Q = Q_\mathrm{p} + Q_s \tag{22}$$

where the subscript "s" denotes a state (solution), the stability of which is in question. Inserting Eq. (22) into Eqs. (1)–(7) and taking into account that solution "s" satisfies equations of Section 3 yield

$$\gamma_\mathrm{p}' = -M_\mathrm{p}, \quad M_\mathrm{p}' = Q_\mathrm{p}$$

$$Q_\mathrm{p}' + \Gamma N_\mathrm{p} - \left(N_\mathrm{p} + C - \Gamma^2/2\right)M_\mathrm{p} + S_\mathrm{p} = 0$$

$$N_\mathrm{p}' - \Gamma Q_\mathrm{p} + \left(Q_\mathrm{p} - \Gamma'\right)M_\mathrm{p} + T_\mathrm{p} = 0$$

$$S'_{\rm p} + (\Gamma - M_{\rm p})T_{\rm p} + \lambda^2 \ddot{\gamma}_{\rm p} = 0$$

$$T'_{\rm p} - (\Gamma - M_{\rm p})S_{\rm p} - \lambda^2 \dot{\gamma}_{\rm p}^2 = 0 \tag{23}$$

Next, neglect the nonlinear terms of perturbations and let for any function

$$Z_{\rm p}(\varphi, t) = Z(\varphi)\exp(i\Omega t)$$

The variables (φ, t) are separated and one obtains the eigenvalue problem for the system

$$\gamma' = -M, \quad M' = Q$$

$$Q' + \Gamma N - (C - \Gamma^2/2)M + S = 0$$

$$N' - \Gamma Q - \Gamma' M + T = 0$$

$$S' + \Gamma T - \omega^2 \gamma = 0$$

$$T' - \Gamma S = 0 \tag{24}$$

where $\omega = \lambda\Omega$ is the dimensionless frequency of vibration. The varying coefficients in Eq. (24) depend on function $\Gamma = \Gamma(\varphi)$ which is found from Eqs. (18)–(21) and represents the equilibrium stability of which is in question.

The same procedure yields from Eq. (15) in the linear approximation

$$S = \omega^2(w + v_{\rm s}\gamma), \qquad T = \omega^2(v - w_{\rm s}\gamma) \tag{25}$$

The amplitudes of vibration in Eqs. (24) and (25) should not be confused with the variables in Eqs. (1)–(7) although the same notations were retained. The system (24) must be supplemented by six appropriate homogeneous boundary conditions.

Applying for the solution of Eq. (24) the finite element method, each variable is approximated by the interpolation function

$$z(x) = n1(x)z_i + n2(x)z_{i+1}$$

$$n1(x) = \frac{x_{i+1} - x}{\delta}, \quad n2(x) = \frac{x - x_i}{\delta}, \quad \delta = x_{i+1} - x_i \tag{26}$$

Next, each Eq. (24) is integrated over the length of an element with the weighting function equal to unity inside the element and zero elsewhere. This approach does not require the differentiation of the assumed interpolation functions and considerably simplifies the element assembly procedure (at the possible cost of increased number of elements). Dividing the total length into n_0 equal elements, the discrete system is obtained

$$\gamma_{i+1} - \gamma_i + n(M_{i+1} + M_i) = 0, \quad M_{i+1} - M_i - n(Q_{i+1} + Q_i) = 0$$

$$Q_{i+1} - Q_i + N_i G1_i + N_{i+1} G2_i - M_i H1_i - M_{i+1} H2_i + n(S_{i+1} + S_i) = 0$$

$$N_{i+1} - N_i - Q_i G1_i - Q_{i+1} G2_i - M_i F1_i - M_{i+1} F2_i + n(T_{i+1} + T_i) = 0$$

$$S_{i+1} - S_i + T_i G1_i + T_{i+1} G2_i - \omega^2 n(\gamma_{i+1} + \gamma_i) = 0$$

$$T_{i+1} - T_i - S_i G1_i - S_{i+1} G2_i = 0 \tag{27}$$

where

$$n = \int_{x_i}^{x_{i+1}} n1(x)\,\mathrm{d}x = \int_{x_i}^{x_{i+1}} n2(x)\,\mathrm{d}x = \delta/2$$

$$F1_i = \int_{x_i}^{x_{i+1}} n1(x)\Gamma'(x)\,\mathrm{d}x, \quad G1_i = \int_{x_j}^{x_{i+1}} n1(x)\Gamma(x)\,\mathrm{d}x,$$

$$H1_i = \int_{x_i}^{x_{i+1}} n1(x)\big(C - \Gamma^2(x)/2\big)\,\mathrm{d}x$$

$$x_i = (i-1)\delta, \quad i = 1, 2, \ldots, n_0 \tag{28}$$

Similar formulas apply for $F2_i$, $G2_i$, $H2_i$ of the ith element. With an introduction of the vector-column

$$Z_i = (\gamma_i, M_i, Q_i, N_i, S_i, T_i)^{\mathrm{T}} \tag{29}$$

the matrix for the start of an element BS_i, and the matrix for the end of an element BE_i

$$BS_i = \begin{bmatrix} -1 & n & 0 & 0 & 0 & 0 \\ 0 & -1 & -n & 0 & 0 & 0 \\ 0 & -H1_i & -1 & G1_i & n & 0 \\ 0 & -F1_i & -G1_i & -1 & 0 & n \\ -n\omega^2 & 0 & 0 & 0 & -1 & G1_i \\ 0 & 0 & 0 & 0 & -G1_i & -1 \end{bmatrix}$$
$$BE_i = \begin{bmatrix} 1 & n & 0 & 0 & 0 & 0 \\ 0 & 1 & -n & 0 & 0 & 0 \\ 0 & -H2_i & 1 & G2_i & n & 0 \\ 0 & -F2_i & -G2_i & 1 & 0 & n \\ -n\omega^2 & 0 & 0 & 0 & 1 & G2_i \\ 0 & 0 & 0 & 0 & -G2_i & 1 \end{bmatrix} \tag{30}$$

the system (27) can be written as

$$BS_i Z_i + BE_i Z_{i+1} = 0, \quad i = 1, 2, \ldots, n_0 \tag{31}$$

The assembly procedure and incorporation of the boundary conditions will be shown below in the numerical examples.

5. Numerical examples

5.1. Example 1

This example concerns the famous problem of a cantilever beam under a compressing follower force (Beck, 1952; Bolotin, 1963) and serves to check the accuracy of the subdomain collocation against an exact solution. The stability of a straight configuration of the beam is governed by

$$w'''' + pw'' - \omega^2 w = 0$$

$$w(0) = w'(0) = 0$$

$$w''(1) = w'''(1) = 0 \tag{32}$$

where (p, ω) are the dimensionless force and frequency. Introduction of the rotation β, the bending moment M, and the shear force Q reduces the problem to

$$w' = \beta, \quad \beta' = M, \quad M' = Q, \quad Q' + pM - \omega^2 w = 0 \tag{33}$$

$$w(0) = \beta(0) = 0, \quad M(1) = Q(1) = 0 \tag{34}$$

Using the subdomain collocation method with the interpolation functions (26) for each variable and integrating each Eq. (33) over the element length, one obtains

$$w_i - w_{i+1} + n(\beta_i + \beta_{i+1}) = 0$$

$$\beta_i - \beta_{i+1} + n(M_i + M_{i+1}) = 0$$

$$M_i - M_{i+1} + n(Q_i + Q_{i+1}) = 0$$

$$Q_{i+1} - Q_i + pn(M_i + M_{i+1}) - \omega^2 n(w_i + w_{i+1}) = 0 \tag{35}$$

Note, that we do not have to replace the derivatives in Eq. (33) by the derivatives of the interpolation functions, but if we did, the result after integration would be the same.

Introducing the vector-column

$$Z_i = (w_i, \beta_i, M_i, Q_i)^{\mathrm{T}} \tag{36}$$

and the matrixes BS for the start of an element, and BE for the end

$$BS = \begin{bmatrix} 1 & n & 0 & 0 \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & n \\ -n\omega^2 & 0 & pn & -1 \end{bmatrix} \qquad BE = \begin{bmatrix} -1 & n & 0 & 0 \\ 0 & -1 & n & 0 \\ 0 & 0 & -1 & n \\ -n\omega^2 & 0 & pn & 1 \end{bmatrix} \tag{37}$$

Eq. (35) can be written as

$$BS\,Z_i + BE\,Z_{i+1} = 0 \tag{38}$$

Now the eigenvalues are to be found from (assuming the total of four elements but the pattern is easily seen for any number)

$$\det\begin{bmatrix} bs & BE & B0 & B0 & b0 \\ b0 & BS & BE & B0 & b0 \\ b0 & B0 & BS & BE & b0 \\ b0 & B0 & B0 & BS & be \end{bmatrix} = 0 \tag{39}$$

Here matrixes

$$bs = \begin{bmatrix} 0 & 0 \\ n & 0 \\ 1 & n \\ pn & -1 \end{bmatrix}, \qquad be = \begin{bmatrix} -1 & n \\ 0 & -1 \\ 0 & 0 \\ -n\omega^2 & 0 \end{bmatrix} \tag{40}$$

are obtained by deleting the first two columns in BS and the last two columns in BE, respectively. This reflects the imposition of the boundary conditions (34). Matrixes $b0$, $B0$ are 2×4 and 4×4 zero matrixes, respectively. For another set of the boundary conditions only the matrixes bs, be are to be changed while the rest of the Eq. (39) remains the same.

The exact frequencies for the considered problem can be found from

$$p^2 + p\omega \sin(r_1)\sinh(r_2) + 2\omega^2\left[1 + \cos(r_1)\cosh(r_2)\right] = 0$$

$$r_1^2 = \sqrt{(p/2)^2 + \omega^2} + p/2, \qquad r_2^2 = \sqrt{(p/2)^2 + \omega^2} - p/2 \tag{41}$$

When $p = 0$ the system is conservative and consequently all roots of (39) and (41) are real numbers. When the load increases the two lowest frequencies move closer and the load at which they coincide is the critical load p_{cr} (flutter instability). In Table 1 the finite element results are compared to the exact solution. For 15 elements the critical load differs from the exact value by 3%.

Eqs. (1)–(7) do not account for damping and some comments about its influence in the presence of nonconservative forces are here in order. To analyze stability, the equations of perturbed equilibrium were linearized and Liapunov method of the first approximation was applied. It was found that for $p < 20$ all characteristic numbers $q = i\omega$ are pure imaginary. In accordance with Liapunov, when real parts of characteristic numbers of the linearized equations are zero these equations cannot serve to establish the stability or instability of the equilibrium. It is well known also that for the conservative system with the pure imaginary characteristic numbers, an addition of a small damping produces the negative real

Table 1
Eigenvalues and critical load for Beck (1952) problem

p	Exact		10 elements		15 elements	
	ω_1	ω_2	ω_1	ω_2	ω_1	ω_2
0	3.52	22.0	3.53	22.9	3.52	22.4
10	5.18	18.6	5.20	19.6	5.19	19.0
20	10.5	11.5	9.19	14.1	9.65	12.9
p_{cr}	20.05		21.2		20.6	

parts and stabilizes the system while in the presence of nonconservative forces this is not so. For the review of an early development see Herrmann (1967).

Two approaches are possible to clarify the situation. The first relies on the nonlinear equations of the perturbed equilibrium and allows for the global dynamic analysis. For the discrete system this approach was applied by Kounadis (1992) but the nonlinear partial differential equations (23) are even more difficult for analysis.

The second approach is to introduce damping in the linear equation. Doing so for the column under consideration, Bolotin and Zhinzher (1969) showed that, if the column is made of the standard viscoelastic material (internal damping only), the critical load for the infinitesimal damping equals to 10.94. Denisov and Novikov (1975) considered both the internal and external damping and came to the conclusion that depending on their ratio, when both are small, the critical load can be anywhere between 20.05 and 10.94.

The damping could be accounted for in the equations of motion (1)–(7) but this analysis is beyond the scope of the present paper.

5.2. Example 2

Circular arch fixed at one end and loaded by a normal inward follower force at the other end. This is the problem B13 of Vitaliani et al. (1997), where both the equilibrium position and frequencies of vibration in the vicinity of this position were determined by the finite element method.

To find the equilibrium configuration analytically the solution (Eq. (19)) of Eq. (17) is used. The boundary conditions at the loaded end

$$M(0) = 1 - \Gamma(0) = 0, \quad N(0) = C - \Gamma^2(0)/2 = 0, \quad Q(0) = -\Gamma'(0) = -p \tag{42}$$

yield three equations

$$\Gamma_0 \mathrm{sn}\,\tau = \mathrm{dn}\,\tau, \quad 2(2k^2 - 1)h^2 = 1, \quad h\Gamma_0 \mathrm{cn}\,\tau = p\,\mathrm{dn}^2\tau \tag{43}$$

from which after some manipulations one finds

$$h^2 = p, \quad k^2 = \frac{2p+1}{4p}, \quad \mathrm{sn}^2\tau = \frac{2}{1+2p}, \quad \Gamma_0^2 = p - \frac{1}{4p} \tag{44}$$

From the requirement $k^2 < 1$ it follows that this solution is valid for $p > 1/2$. Otherwise the solution (21) is used:

$$\Gamma = \Gamma_0 \mathrm{dn}^{-1}(h\varphi + \tau), \quad \Gamma_0^2 = \mathrm{dn}^2\tau = 1 - 2p, \quad k^2 = \frac{4p}{2p+1}, \quad h^2 = (2p+1)/4 \tag{45}$$

The displacements could be found from Eqs. (11) and (12) for the equilibrium state and from Eq. (25) for the state of vibration. However, there is no need to do so if one is interested in the stability of equilibrium only.

Now all the data are available to write the eigenvalues determinant

$$\det \begin{bmatrix} bs_1 & BE_1 & B0 & B0 & b0 \\ b0 & BS_2 & BE_2 & B0 & b0 \\ b0 & B0 & BS_3 & BE_3 & b0 \\ b0 & B0 & B0 & BS_4 & be_4 \end{bmatrix} = 0 \tag{46}$$

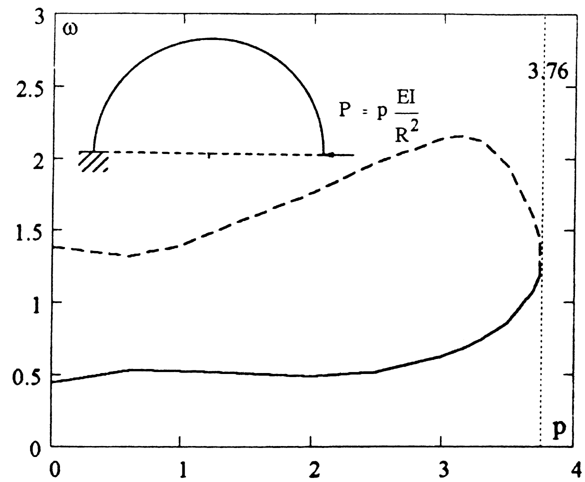

Fig. 2. First and second eigenvalues of the loaded arch.

where matrixes

$$bs_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \\ -n\omega^2 & -1 & G1_1 \\ 0 & -G1_1 & -1 \end{bmatrix}, \qquad be_4 = \begin{bmatrix} n & 0 & 0 \\ 1 & -n & 0 \\ -H2_4 & 1 & G2_4 \\ -F2_4 & -G2_4 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{47}$$

reflect the boundary conditions

$$M(0) = Q(0) = N(0) = 0, \qquad \gamma(\alpha) = S(\alpha) = T(\alpha) = 0 \tag{48}$$

and α is the arch subtended angle. The matrixes *b0*, *B0* are 3×6 and 6×6 zero matrixes, respectively. Eq. (46) is again written for four elements but it clearly shows the pattern for any number of elements. The calculated first and second frequencies for $\alpha = \pi$ and 15 elements are shown in Fig. 2. The critical load $p_{cr} = 3.76$ compares well with the value 3.65 (dimensionless) found by Vitaliani et al. (1997). In accordance with the latter work the slight Rayleigh-type damping reduces the critical load by 22%.

Omitting the details we notice in passing that if the load is directed outward the critical load decreases to 1.41.

6. Conclusion

For the nonlinear equilibrium of uniform beams and circular arches under nonconservative loading, the closed form solution is obtained.

The estimation of the critical load is reduced to an eigenvalue problem for the sixth order system of the linear differential equations with varying coefficients. To compute the eigenvalues from this system the subdomain collocation method and the finite element approach are used. The critical load, computed for the cantilever semicircular arch, is in close agreement with that obtained by Vitaliani et al. (1997).

The method is quite accurate and yields a simple assembled matrix. Its application is limited to the plane deformation of beams and circular arches with an uniform cross section.

References

Beck, M., 1952. Die Knicklast des einseitig eingespannten, tangential gedrückten Stabes. Zeitschrift für angewandte Mathematik und Physik 3, 225–228.

Bolotin, V.V., 1963. Nonconservative Problems of the Theory of Elastic Stability. Pergamon Press, Oxford (English translation edited by G. Herrmann).

Bolotin, V.V., Zhinzher, N.I., 1969. Effect of damping on stability of elastic systems subjected to nonconservative forces. Int. J. Solids Structures 5, 965–989.

Chernykh, K.F., 1986. Non-Linear Theory of Elasticity (in Russian). Mashinostroenie.

Denisov, G.G., Novikov, V.V., 1975. On the stability of a rod under follower force (in Russian). Mekhanika Tverdogo Tela 1, 150–154.

Detinko, F.M., 1998. Circular arch with no instability. Int. J. Solids Structures 35, 3213–3220.

Herrmann, G., 1967. Stability of equilibrium of elastic systems subjected to nonconservative forces. AMR 20, 103–108.

Kounadis, A.N., 1992. On the paradox of the destabilizing effect of damping in non-conservative systems. Int. J. Non-Linear Mechanics 27, 597–609.

Simmonds, J.G., 1979. Accurate nonlinear equations and a perturbation solution for the free vibrations of a circular elastic ring. Journal of Applied Mechanics 46, 156–160.

Vitaliani, R.V., Gasparini, A.M., Saetta, A.V., 1997. Finite element solution of the stability problem for nonlinear undamped and damped systems under nonconservative loading. Int. J. Solids Structures 34, 2497–2516.

Zienkiewicz, O.C., Taylor, R.L., 1989. The Finite Element Method. McGraw-Hill, New York.